Last class:

Theorem (Abel) Assume $f(x) = \sum a_n x^n$ has radius of convergence R

If power series converges at $x = R \Rightarrow$ its value is $\lim_{x \to R} f(x)$

" " " " $x = -R \Rightarrow$ " " " $\lim_{x \to -R} f(x)$

Proof. Case 1 $R = 1$, series converges at $x = 1$

also assumed $f(1) = \sum_{k=0}^{\infty} a_k = 0$

$\Rightarrow$ If $S_n = \sum_{k=0}^{n} a_k$ then $\boxed{\lim_{n \to \infty} S_n = 0}$

statement says: $f(x)$ is cont. at $x = 1$

enough to show: If $f_n(x) = \sum_{k=0}^{n} a_k x^k$

$\Rightarrow (f_n) \longrightarrow f$ uniformly on $[0,1]$.

$\Leftarrow$ (f_n) uniformly Cauchy.

need to estimate (for $n > m$)

$$|f_n(x) - f_m(x)| = \left| \sum_{k=m+1}^{n} a_k x^k \right|$$

(last time)

$$\leq \cdots\cdots$$

$$\leq \left| \sum_{k=m+1}^{n-1} (1-x) s_k x^k \right| + |s_n x^n| + |s_{m+1} x^m|$$

$< \varepsilon/3$ $\quad$ $< \varepsilon/3$ $\quad$ $< \varepsilon/3$

want to use $\varepsilon/3$ argument, i.e.

$x \in [0,1] \Rightarrow |x| \leq 1$

$\lim_{n\to\infty} s_n \to 0 \Rightarrow \exists N$ s.t. $|s_n| < \varepsilon/3$ for all $n > N$

assume $n > N$, $m > N$ $\Rightarrow$

$$|s_n x^n| \leq |s_n| < \varepsilon/3$$

$$|s_{m+1} x^m| \leq |s_{m+1}| < \varepsilon/3$$

For first summand, we also have $|s_k| < \varepsilon/3$, $m+1 \leq k \leq n-1$

$$\Rightarrow \left| \sum_{k=m+1}^{n-1} (1-x)\delta_k x^k \right| < \frac{\varepsilon}{3} \underbrace{\left| \sum_{k=m+1}^{n-1} (1-x) x^k \right|}_{\text{use formula for geometric sum}}$$

$$\leq \frac{\varepsilon}{3} \left| \cancel{(1-x)} \frac{x^{m+1} - x^n}{\cancel{(1-x)}} \right|$$

use $|x^{m+1} - x^n| \leq |x^{m+1}|$ because $0 \leq x \leq 1$.

$$\leq \frac{\varepsilon}{3} |x^{m+1}| \leq \frac{\varepsilon}{3}$$

have shown:

$$|f_n(x) - f_m(x)| < \varepsilon$$

for all $x \in [0,1]$

for all $n, m > N$.

$\Rightarrow$ $(f_n) \rightarrow f$ uniformly on $[0,1]$

$\Rightarrow$ f continuous $\Rightarrow$ claim ✓

Case 2 assume now arbitrary radius of convergence $R > 0$
and $\sum a_n x^n$ converges for $x = R$

can be reduced to case 1 via simple coordinate transf.

consider the function $g(x) = f(Rx) = \sum_{k=0}^{\infty} a_k R^k x^k$

check: radius of convergence of is equal to 1

and $g(1) = \sum a_k R^k$ does converge by our ass.

$\Rightarrow$ g satisfies cond. in case 1

$\Rightarrow$ g is continuous by case 1

$\Rightarrow$ $f(x) = g(x/R)$ is also continuous

$\Rightarrow$ claim.

radius of conv. $= 1$

method 1 if $|x| < 1 \Rightarrow |Rx| < R \Rightarrow \sum a_k (Rx)^k$ does converge

method 2 if $|x| > 1$ $|Rx| > R \Rightarrow$ " " does NOT converge.

see next page

Case 3. $f(x) = \sum a_n x^n$ has radius of convergence R

and series converges for $x = -R$

Consider $h(x) = f(-x) = \underbrace{\sum_{n=0}^{\infty} a_n (-1)^n x^n}_{\text{series converges for } x = R}$

$\Rightarrow$ h continuous by case 2

$\Rightarrow$ f continuous.

method 2 from previous page:

let $\tilde{R}$ be radius of conv. of $\sum a_n R^n x^n$

$$\Rightarrow \frac{1}{\tilde{R}} = \limsup |R^n a_n|^{1/n} = \limsup R |a_n|^{1/n}$$

$$= R \cdot \frac{1}{R} = 1$$

$\frac{1}{R} = \limsup |a_n|^{1/n}$

Remark: (1) Have seen: If a power series $\sum a_n x^n$
has radius of convergence $R > 0$

$\Rightarrow$ get function $f(x) = \sum a_n x^n$ for $|x| < R$
which is differentiable

derivative $f'(x) = \sum a_n n x^{n-1}$
again has radius of conv. R

$\Rightarrow$ again differentiable for $|x| < R$

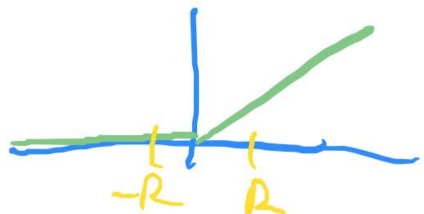
$\Rightarrow$ get functions which are ∞ times differentiable.

(2) See later: power series of $f(x)$ = its Taylor series!

(3) not every cont. function can be approx. by a power series with positive radius of convergence:

reason: not every cont. function is differentiable

e.g.



$f(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$

not differentiable at $x=0$

$\Rightarrow$ $f(x)$ can not be approx. via power series near $x=0$ with radius of conv. $R > 0$

(4) There exists the famous Weierstrass approximation theorem which says:

For any continuous function $f(x)$ on an interval $[a,b]$ there exists a sequence of polynomials $(p_n(x))$ such that

$p_n \to f$ uniformly.

(see book Chapter 27)